# Energy-Like Lyapunov Functions for Power System Stability Analysis

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*Abstract* —**In this paper, a concept of strong stability of an equilibrium point of an electric power system is introduced. It is shown that almost all stable equilibria of the standard transient stability model are strongly stable and that strong stability is a necessary and sufficient condition for the existence of a local energy-like Lyapunov function for all small perturbations of the nominal system. Such a Lyapunov function is explicitly constructed. A complete local analysis of the stability of power system equilibria in the presence of transfer conductances is given.**

## I. INTRODUCTION

EFFICIENT analytical methods for the assessment of power system stability are actively being sought by industry. A great deal of effort has been devoted to the application of Lyapunov methods to this problem beginning with the work of Gless [1], and El-Abiad and Nagappan [2] and continuing to the present time (see, for example, the text by Pai [3] for a recent survey). The use of energy functions in constructing Lyapunov functions is especially appealing because of the physical insight provided by them, and also because classical results obtained using energy-related Lyapunov function, such as the theorems of Lagrange and Ĉetaev [4], are among the sharpest available.

The application of energy methods to power system stability analysis preceeds the introduction of Lyapunov techniques and begins with the papers of Magnusson [5] and Aylett [6]. The interest in energy methods was recently revived by Athay *et al.* [7] with the introduction of an energy-like Lyapunov function. Their promising results were followed by the introduction of many other energy-related Lyapunov functions. Among these, we note those of Bergen and Hill [8], Athay and Sun [9], Michel *et al.* [10], and Narasimhamurthi and Musavi [11].

The major variant among these recent investigations [7]–[11] is the treatment of system loads. It is well known that load is a critical factor in power system stability. Thus it is most distressing that even the most elementary nontrivial load model, constant impedance to ground, has defied rigorous analysis using energy methods. The basic difficulty is that the incorporation of constant impedance loads with the usual generator and transmission-line models employed in transient stability analysis leads to a reduced bus admittance matrix with transfer conductances. The appropriate definition of energy functions in the presence of transfer conductances is not at all clear.

In fact, in a recent paper [12], Narasimhamurthi claims that no smooth modification of the lossless system transient energy function to accommodate line losses (however small) can lead to a Lyapunov function. He carefully points out that his result does not mean that a Lyapunov function does not exist for a system with transfer conductances, but, if it does it must be substantially different from an energy function.

In this paper, we provide a complete local analysis of the stability of power system equilibria in the presence of transfer conductances. We will show that the conclusions of [12] are too pessimistic and that a local energy-like Lyapunov function does exist, in general, for stable equilibria of systems with transfer conductances. In doing so, we will introduce the notion of *strong stability* of an equilibrium point of a power system. Roughly speaking, an equilibrium point is strongly stable if it is stable and remains stable under sufficiently small arbitrary perturbations of the reduced bus admittance matrix. It will be shown that almost all stable equilibria are strongly stable, and a simple characterization of the exceptional ones will be given. Our key result is that strong stability is a necessary and sufficient condition for the existence of a (local) energy-like Lyapunov function for arbitrary small perturbations of the nominal system. We give an explicit construction for such a Lyapunov function.

Because our results are local in character, they serve only to determine the stability of the equilibrium point and cannot be directly applied to determine the domain of stability. Nevertheless, our results suggest that there is good reason to believe that energy functions can play a useful role in resolving this problem.

Transfer conductances give rise to nonconservative forces which are analogous to forces in mechanical systems called circulatory forces. The stability of mechanical systems under the influences of circulatory forces has been studied, most notably by Huseyin [13] and Leipholz [14]. Our point of view has been strongly influenced by these investigators and the text [13] provides excellent background for our analysis. The search for energy-like Lyapunov functions is closely related to questions of the existence of variational principles which produce a given set of differential equa-

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tions. For an investigation of such issues in the spirit of the analysis provided here the reader is referred to Kwatny *et al.* [15] and Bahar and Kwatny [16].

## II. Modeling and Definitions

### *Basic Assumptions*

The usual model of an electric power system used in analytical studies of power system transient stability [1]–[3], [7]–[11], [21] is based on several fundamental assumptions, the most important of which are as follows.

1) Each synchronous generator may be represented by a constant voltage behind its transient reactance.
2) Mechanical power delivered to each turbine generator rotor is constant.
3) Transmission lines may be modeled by a synchronous admittance (usually lossless).
4) Loads may be characterized by a constant synchronous admittance to ground.

Certainly, the most controversial aspect of the above assumptions is the load model. Constant admittance loads are perhaps the simplest nontrivial load model one can devise. Nevertheless, the resulting stability problem is subtle and, as yet, unsolved. Most analyses add the further restriction that the reduced bus admittance matrix be free of transfer conductances. This is tantamount to the assumption of lossless transmission lines and essentially restricts lossy loads to generator busses. Other models have also been used to circumvent the difficulties associated with constant admittance loads. These include constant power [8] and constant real power with voltage dependent reactive power [11].

It should be noted that the addition of constant power loads does not in any substantial way complicate the ideal analysis of a system with constant admittance loads. A mix of constant power and constant admittance loads is probably a reasonable first approximation in the majority of situations. It is essential that any theory of power system transient stability be capable of dealing with constant admittance loads.

The chief deficiency of constant power and constant admittance loads is that they do not reflect the frequency dependence of load. To some extent, this is remedied by replacing the constant power assumption by constant torque as in [9]. It is generally believed that for the transient stability problem, load frequency effects are not of first-order significance. Nevertheless, even small frequency dependences have the potential to substantially affect power system stability. We will comment further on this point later.

The generator model is another factor of some concern. It is well known that the constant voltage behind transient reactance model follows from a simplification of Park's equations based on the assumption of balanced operation, constant field flux, and fast asymptotically stable electrical transients. The resultant model is a single "swing equation" in torque form which is usually converted to power form. The conversion involves approximating certain rotor speed dependent terms by replacing actual speed by synchronous speed. It is the failure to account for small speed perturbation terms which has generated serious objections [22].

### *The Standard Model*

The assumptions delineated above lead to what we will call the standard model for power system transient stability analysis. Consider a system consisting of $n$ generators. The equations of motion are

$$M_i\ddot{\delta}_i + D_i\dot{\delta} = P_i - P_{ei}, \qquad i = 1, 2, \cdots, n \tag{2.1}$$

where

$$P_{ei} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left[F_{ij}\sin(\delta_i - \delta_j) + H_{ij}\cos(\delta_i - \delta_j)\right] \tag{2.2}$$

$$P_i = P_{mi} - E_i^2 G_{ii} \tag{2.3}$$

$$F_{ij} = E_i E_j B_{ij}, \quad H_{ij} = E_i E_j G_{ij} \tag{2.4}$$

and where for the unit designated by $i$, the following nomenclature is employed:

| | |
|---|---|
| $P_{mi}$ | mechanical power, |
| $G_{ii}$ | driving point conductance, |
| $E_i$ | constant voltage behind direct axis transient reactance, |
| $\delta_i$ | generator angle deviation from synchronous reference, |
| $M_i$ | moment of inertia, |
| $D_i$ | damping constant, |
| $B_{ij}(G_{ij})$ | transfer susceptance (conductance) in the reduced bus admittance matrix, |

where, in the absence of phase shifting transformers, $B_{ij}, G_{ij}$ and hence $F_{ij}, H_{ij}$ are symmetric.

The equations can be written in the vector form

$$M\ddot{\delta} + D\dot{\delta} + f(\delta) = P \tag{2.5}$$

where

$$M = \operatorname{diag}(M_i) \tag{2.6}$$

$$D = \operatorname{diag}(D_i) \tag{2.7}$$

$$f_i(\delta) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left[F_{ij}\sin(\delta_i - \delta_j) + H_{ij}\cos(\delta_i - \delta_j)\right]. \tag{2.8}$$

There are several important properties associated with $f(\delta)$. Obviously, $f(\delta)$ is $2\pi$-periodic in each of the variables $\delta_i$. Also $f(\delta)$ has a translational symmetry. That is, $f(\delta)$ is invariant under a uniform translation of the angles

$$\delta_i \to \delta_i + c, \qquad \text{for } i = 1, \cdots, n \tag{2.9}$$

where $c$ is a constant. Another way of stating this is as follows. Let $\mathbf{1}$ be the $n$-vector

$$\mathbf{1} \triangleq [1, \cdots, 1]'. \tag{2.10}$$

Then

$$f(\delta)=f(\delta+c\mathbf{1}) \tag{2.11}$$

for any constant $c$.

It is easy to overlook the significance of the translational symmetry because it has such an obvious physical interpretation. It means simply that only the relative motions of the angular displacements are unique. Thus any equilibrium point of interest is actually a point in a one-dimensional manifold of equilibria in the $2n$-dimensional state space and it only makes sense to discuss the stability of the entire manifold. The usual remedy is to measure displacement relative to an arbitrary selected swing bus or to define the so-called center of angle coordinates. In any case, the state space is reduced to dimension $2n-1$ and the equilibrium manifold is collapsed to a point.

When the system is conservative, i.e., in the absence of damping and transfer conductances, the translational symmetry is directly associated with a conservation law or first integral: total angular momentum is constant. Thus a second reduction is obtainable so that a reduction of the state space to dimension $2n-2$ can be achieved [23].

As it turns out, this reduction is not restricted to purely conservative power systems. It is known (Willems [21]) that it works when uniform damping is present (in the absence of transfer conductances), an often used approximation. It does not work, however, when arbitrary damping is present. The fact that various dimensions for the state space have been employed in the literature has sometimes made comparison between methods difficult. Willems [21] describes the situation very well and builds a case for conducting the analysis in the $2n$-dimensional state space.

Another important fact is that under certain circumstances the function $f(\delta)$ is derivable from a potential function. That is, there exists a scalar function $U(\delta)$, such that

$$f(\delta)=\frac{\partial U}{\partial \delta}. \tag{2.12}$$

It is important to know when this is the case. As is well known, a function $f(\delta)$ is (directly) derivable from a potential if and only if its Jacobian is symmetric, i.e.,

$$\frac{\partial f}{\partial \delta}=\left[\frac{\partial f}{\partial \delta}\right]'. \tag{2.13}$$

We can easily compute the Jacobian using (2.8):

$$\left[\frac{\partial f}{\partial \delta}\right]_{ij}=\begin{cases}-F_{ij}\cos(\delta_i-\delta_j)+H_{ij}\sin(\delta_i-\delta_j), & i\neq j\\ \sum\limits_{\substack{l=1\\ l\neq i}}^{n}\left[F_{il}\cos(\delta_i-\delta_l)-H_{ij}\sin(\delta_i-\delta_l)\right], & i=j.\end{cases} \tag{2.14}$$

Moreover, the Jacobian can be separated into a symmetric part $(\partial f/\partial \delta)_s$ plus an antisymmetric part $(\partial f/\partial \delta)_a$ as follows:

$$\left[\left(\frac{\partial f}{\partial \delta}\right)_s\right]_{ij}=\begin{cases}-F_{ij}\cos(\delta_i-\delta_j), & i\neq j\\ \sum\limits_{\substack{l=1\\ l\neq i}}^{n}\left[F_{il}\cos(\delta_i-\delta_l)-H_{il}\sin(\delta_i-\delta_l)\right], & i=j\end{cases} \tag{2.15}$$

$$\left[\left(\frac{\partial f}{\partial \delta}\right)_a\right]_{ij}=H_{ij}\sin(\delta_i-\delta_j). \tag{2.16}$$

Clearly, if there are no transfer conductances, then $H_{ij}=0$ and the Jacobian is symmetric. Thus $f(\delta)$ is derivable from a potential function, $U(\delta)$, and $U(\delta)$ is commonly interpreted as the system potential energy. The fact that in the presence of transfer conductances $f(\delta)$ is not directly integrable is the essential difficulty in analyzing such systems.

The asymmetry in the Jacobian of $f(\delta)$ when transfer conductances are present means that $f(\delta)$ includes non-conservative forces of a type called circulatory forces in mechanics. Circulatory forces introduce effects that are quite different from dissipative forces and interact with dissipative forces in ways which are not intuitively obvious. In the absence of transfer conductances and dissipation the equations of motion (2.5) are conservative.

*Stability of Equilibria*

An equilibrium of (2.5) is a point $\delta^*\in R^n$ such that $\dot{\delta}^*\equiv 0$. Thus equilibria are roots of the equation

$$f(\delta)=P. \tag{2.17}$$

Let $\delta^*$ be a solution of (2.17). Then because of the translational symmetry of $f(\delta)$ all points of the type

$$\delta=\delta^*+c\mathbf{1} \tag{2.18}$$

with $c$ an arbitrary real constant are also equilibria. Consequently, any equilibrium point belongs to a one-dimensional manifold of equilibrium points. Consider the equilibrium manifold $M$ in $R^n$ associated with the equilibrium point $\delta^*$.

$$M=\delta^*+\operatorname{span}\{\mathbf{1}\}. \tag{2.19}$$

In the $2n$-dimensional state space composed of points $(\omega,\delta)$, the one-dimensional equilibrium manifold is the manifold

$$\hat{M}=\{(\omega,\delta)\mid\omega=0,\ \delta\in M\}. \tag{2.20}$$

We are interested in the stability of $\hat{M}$ and we use the usual definition of stability for an invariant set. Recall, that for any set $\hat{M}$ in $R^{2n}$, an $\eta$-neighborhood $U_\eta(\hat{M})$ is the set of $y$ in $R^{2n}$ such that $\operatorname{dist}(y,\hat{M})<\eta$, [17].

*Definition:* An invariant set $\hat{M}$ of (2.5) is *stable* if for any $\epsilon>0$ there is an $\eta>0$ such that for any initial $(\omega^\circ,\delta^\circ)$ in $U_\eta(\hat{M})$, the corresponding solution $(\omega(t),\delta(t))$ is in

$U_\epsilon(\hat{M})$ for $t \geqslant 0$. $\hat{M}$ is *asymptotically stable* if it is stable and in addition each solution with initial state in $U_\eta(\hat{M})$ approaches $\hat{M}$ as $t \to \infty$.

We will state two basic theorems which characterize the stability of invariant sets in terms of Lyapunov functions. First consider an autonomous system on an $m$-dimensional state space, and suppose that $\Omega$ is a $p$-dimensional invariant set. Furthermore, suppose that $\Omega$ can be characterized in the following way. There exists a continuous function $g$: $R^m \to R^{m-p}$ such that

$$\Omega = \{ y \in R^m | g(y) = 0 \}. \tag{2.21}$$

Clearly, this is the situation of interest in power system stability, in which case $m = 2n$, $p = 1$, and the map $g$ is linear.

*Definition:* A scalar function $V(y)$ is said to be *positive definite with respect to* $\Omega$ in an open region $U \supset \Omega$ if

1) $V(y)$ and its first partial derivatives are continuous in $U$.
2) $V(y) = 0$ for $y \in \Omega$.
3) $V(y) \geqslant W(g(y))$, where $W(g(y))$ is an ordinary positive definite function on the Image of $U \subset R^p$, under $g$.

If, in addition, $\dot{V} \leqslant 0$ on $U$, $V$ is called a *Lyapunov function* (*with respect to* $\Omega$).

The following theorem is a simple extension of Lyapunov's classic stability theorem.

*Theorem 2.1:* If a Lyapunov function exists in some open neighborhood $U$ of an invariant set $\Omega$, then $\Omega$ is stable.

A more subtle theorem was established by Willems [21] which provides conditions for asymptotic stability of $\Omega$ and also leads to a procedure for estimating the region of attraction for $\Omega$:

*Theorem 2.2:* Suppose there exists a Lyapunov function $V(y)$ on an open region $U \supset \Omega$ such that

1) $V(y) = a$ on the boundary of $U$ and $V(y) < a$ in $U$.
2) $g(y)$ is bounded in $U$.
3) $\dot{V}$ does not vanish identically on any trajectory in $U$ which does not belong entirely in $\Omega$.

Then $\Omega$ is asymptotically stable and every trajectory in $U$ tends to $\Omega$ as $t \to \infty$.

*Load Perturbations and Linearization*

We will be interested in the (local) stability of the equilibrium manifold under small but otherwise arbitrary perturbations in the system loads. This can be translated directly into small, arbitrary perturbations in the system admittance matrix which, in turn, alters the function $f(\delta)$ as well as the power $P$ appearing in (2.5). In the following paragraphs we develop the model to be used in our analysis.

Consider some bounded neighborhood $U$ of the equilibrium point $\delta^*$. Let $h(\delta)$ be any function defined on $U$ with continuous first partial derivatives. We measure $h$ by the norm

$$|h| = \sup_{\delta \in U} |h(\delta)| + \sup_{\delta \in U} \left| \frac{\partial h(\delta)}{\partial \delta} \right|. \tag{2.22}$$

The function $f(\delta)$ can always be written

$$f(\delta) = K(\delta - \delta^*) + g(\delta) \tag{2.23}$$

where

$$K = \frac{\partial f(\delta^*)}{\partial \delta} \tag{2.24}$$

and

$$g(\delta) = f(\delta) - K(\delta - \delta^*). \tag{2.25}$$

To account for perturbation in loads, we replace $f(\delta)$ by $\hat{f}(\delta)$

$$\hat{f}(\delta) = K(\delta - \delta^*) + \hat{g}(\delta) \tag{2.26}$$

where

$$\hat{g}(\delta) = g(\delta) + h(\delta) \tag{2.27}$$

and $h(\delta)$ is any function defined on $U$ with continuous first partial derivatives, with translational symmetry and

$$|h| \leqslant \eta_0. \tag{2.28}$$

In general, $P$ will also, change under load perturbations so that (2.5) is replaced by

$$M\ddot{\delta} + D\dot{\delta} + \hat{f}(\delta) = \hat{P}. \tag{2.29}$$

Equilibria are solutions of

$$\hat{f}(\delta) = \hat{P} \tag{2.30}$$

and a solution exist iff $\hat{P} \in \text{Range}(\hat{f})$. We will assume that there exists a number $\eta_0 > 0$, and $P_m(h)$ so that an equilibrium $\delta_0(h)$ exists with $\delta_0(0) = \delta^*$ for all $h$ satisfying (2.28). Physically this means that the mechanical power can change to accommodate sufficiently small changes in load.

The transformation $\delta = \delta_0 + x$ allows (2.29) to be written

$$M\ddot{x} + D\dot{x} + \left( K + \frac{\partial \hat{g}(\delta_0)}{\partial x} \right) x + \phi(x) = 0 \tag{2.31}$$

where

$$\phi(x) = \hat{g}(\delta_0 + x) - \hat{g}(\delta_0) - \frac{\partial \hat{g}(\delta_0)}{\partial x} x. \tag{2.32}$$

Note that $\phi(x) = o(\|x\|)$ as $\|x\| \to 0$. It follows that a linear approximation is obtained simply by omitting the term $\phi(x)$. Furthermore, this analysis can be conducted at any point of the equilibrium manifold $\hat{M}$ since translational symmetry is imposed on perturbations. Thus the same linear approximation is valid for the entire manifold.

Finally, note that $\partial \hat{g}(\delta_0)/\partial x$ is a small but arbitrary matrix except that it must possess translational symmetry. Thus, we replace it by $\epsilon F$ where $\epsilon$ is a small parameter and $F$ is an arbitrary matrix possessing translational symmetry. Our linear model for study of the (local) stability of equilibria is then

$$M\ddot{x} + D\dot{x} + (K + \epsilon F)x = 0. \tag{2.33}$$

## III. Systems with Small Transfer Conductances

Consider first the conservative system described by the differential equations

$$M\ddot{x} + Ux = 0 \tag{3.1}$$

where

$$M' = M > 0$$
$$U' = U \tag{3.2}$$

and the matrix $U$ has a translational symmetry. We state the following elementary theorem without proof.

*Theorem 3.1:* The equilibrium manifold of the conservative system (3.1) is stable iff $U \geqslant 0$ with precisely one zero eigenvalue.

Let **F** denote the set of $n \times n$ real matrices having the property of translational symmetry and such that $\|F\| \leqslant 1$.

*Definition:* We say that the equilibrium manifold of system (3.1) is *strongly stable*, if there exists an $\epsilon_0 > 0$ such that the perturbed system

$$M\ddot{x} + (U + \epsilon F)x = 0 \tag{3.3}$$

is stable for each $\epsilon$, $|\epsilon| < \epsilon_0$, and each $F \in \mathbf{F}$.

The following theorem characterizes strongly stable systems.

*Theorem 3.2:* The equilibrium manifold of system (3.1) is strongly stable iff $U$ satisfies the conditions of Theorem 1 and in addition, the eigenvalues of $U$ are distinct.

*Proof:* We will demonstrate sufficiency first. Assume that $U$ has distinct, real roots, one of which is zero and the others positive. The zero root persists under perturbations because $F$ as well as $U$ has translational symmetry. Let $\phi(\lambda)$ denote the characteristic polynomial of $U$ and let $\lambda^*$ denote any one of the positive roots of $U$. Then, since $\lambda^*$ is distinct we can

$$\phi(\lambda) = (\lambda - \lambda^*)P(\lambda), \qquad P(\lambda^*) \neq 0.$$

Note that

$$\phi'(\lambda) = P(\lambda) + (\lambda - \lambda^*)P'(\lambda)$$

from which it follows that $\phi'(\lambda^*) \neq 0$. Now, let $\Delta(\lambda, \epsilon)$ denote the characteristic polynomial of $(U + \epsilon F)$. Clearly, $\Delta(\lambda^*,)) = 0$, and $\Delta'(\lambda^*, 0) = \phi'(\lambda^*) \neq 0$. Thus the implicit function theorem guarantees the existence of a unique, real valued, continuous function of $g(\epsilon)$ and a positive number $\epsilon_0$ such that

$$\lambda^* = g(0)$$
$$\Delta(g(\epsilon), \epsilon) = 0, \qquad \text{for } |\epsilon| < \epsilon_0.$$

It follows that for $\epsilon$ sufficiently small, $U + \epsilon F$ has a positive real root close to $\lambda^*$.

To demonstrate necessity, note that since $U$ has independent eigenvectors we need only consider $U$ in the canonical form

$$U = \operatorname{diag}(\lambda_1, \cdots, \lambda_{n-1}, 0).$$

Suppose now, that $U$ has repeated roots, then we can take $\lambda_1 = \lambda_2$, so that

$$U = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_3, \cdots, 0).$$

Now, take $F$ to be

$$F = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, \cdots, 0\right).$$

It is easy to demonstrate that $U + \epsilon F$ has eigenvalues:

$$\lambda_1 + j\epsilon, \lambda_1 - j\epsilon, \lambda_3, \cdots, \lambda_{n-1}, 0.$$

It follows from Theorem 1 that the system is unstable for every $\epsilon > 0$.

Note that the requirements on $U$ for stability of (3.1) admit repeated roots. Strong stability does not allow repeated roots. This can be viewed as a very simple type of resonance exclusion. Our notion of strong stability, tailored for the power system stability problem, may be viewed as a special case of strong stability for linear reciprocal systems as defined by Hale [17]. The functions belonging to the perturbation class used in [17] are periodic in time. As may be expected, the resulting resonance conditions are considerably more complex. In the context of power system stability, the use of time-varying (perhaps even stochastic) perturbations may be appropriate in view of the fact that the reduced bus admittance parameters change with load perturbations. Destabilization of the swing equations under stochastic perturbations has been observed by Loparo and Blankenship [18].

Our subsequent analysis depends on the notion of a symmetrizable matrix.

*Definition:* A real matrix $A$ is *symmetrizable* if it becomes symmetric upon multiplication by a real, symmetric, positive definite matrix, $S$.

We state without proof the following theorem of Taussky [19]:

*Theorem 3.3:* The following properties are equivalent:

- $A' = SAS^{-1}$ with $S' = S > 0$,
- $A$ is similar to a symmetric matrix,
- $A$ is the product of two symmetric matrices, one of which is positive definite,
- $A$ is symmetrizable,
- $A$ has real characteristic roots and a full set of eigenvectors.

The following corollary will prove useful.

*Corollary 3.1:* If $A$ is symmetizable by a matrix $S$ so that $SA = Q$, then $A$ has real characteristic roots and these roots have the same sign as those of $Q$.

*Proof:* That $A$ has real roots follows from Theorem 3. Note that since $S > 0$, we can write $A = S^{-1}Q$ and $S^{-1}$ can be factored $S^{-1} = TT'$. Thus $A = TT'Q$ and

$$T^{-1}AT = T'QT$$

so that $A$ is similar to the symmetric matrix $T'QT$ whose eigenvalues obviously have the same signs as those of $Q$.

Consider the system

$$M\ddot{x} + Kx = 0 \tag{3.4}$$

where $M' = M > 0$, as before, and $K$ is real and has the translation symmetry property. We can characterize the stability of (3.4) as follows.

*Theorem 3.4:* The equilibrium manifold of system (3.4) is stable iff there exists a symmetric, positive definite matrix, $S$, such that $SM^{-1}K$ is symmetric and satisfies the conditions of Theorem 1.

*Proof:* Clearly (3.4) is unstable iff there exists an nontrivial (eigen)vector, $u$, satisfying

$$[M\lambda^2 + K]u = 0$$

with corresponding eigenvalue $\lambda$ having positive real part, or a repeated eigenvalue with zero real part and without a complete set of eigenvectors (we except the double root at the origin associated with the translational symmetry). It is easy to see that the roots of $\det\{M\lambda^2 + K\}$ are distributed symmetrically with respect to both the real and imaginary axes. It follows that the eigenvalues of a stable system must have zero real part and they must be associated with a complete set of eigenvectors (translational symmetry excepted). Moreover, this implies that $M^{-1}K$ must have positive real eigenvalues except for the single zero eigenvalue corresponding to the translational symmetry and a complete set of eigenvectors. It follows from Theorem 3 that $M^{-1}K$ is symmetrizable by a matrix $S$, and from Corollary 3.1 that $SM^{-1}K$ is nonnegative with the only zero eigenvalue corresponding to the translational symmetry. Thus $SM^{-1}K$ satisfies the conditions of Theorem 1.

It follows from Theorems 2 and 4 that if (3.1) is strongly stable, i.e., stable for each $F \in \boldsymbol{F}$ and $\epsilon$, $|\epsilon| < \epsilon_0$, there exists a symmetrizing matrix $S(\epsilon)$ for $M^{-1}(U + \epsilon F)$. We will show that, for each fixed $F$, $S(\epsilon)$ can be chosen so that it is an analytic function of $\epsilon$ in an open neighborhood of $\epsilon = 0$. Then, we will give a formal construction of $S(\epsilon)$ as a power series in $\epsilon$ about the point $\epsilon = 0$.

Let $\Phi(\epsilon) = M^{-1}(U + \epsilon F)$. $S(\epsilon)$ is a symmetrizing matrix for $K(\epsilon)$ iff it is a symmetric, positive definite matrix satisfying

$$\Phi(\epsilon)S(\epsilon) - S(\epsilon)\Phi'(\epsilon) = 0. \tag{3.5}$$

Since $\Phi(\epsilon)$ has a complete set of eigenvectors for $|\epsilon| < \epsilon_0$ we can construct a transformation matrix $T(\epsilon)$ from the eigenvectors of $\Phi(\epsilon)$, normalized to one, so that

$$\Phi(\epsilon) = T(\epsilon)\Lambda(\epsilon)T^{-1}(\epsilon) \tag{3.6}$$

were

$$\Lambda(\epsilon) = \text{diag}(\lambda_i(\epsilon)) \tag{3.7}$$

and $\lambda_i(\epsilon)$ are the eigenvalues of $K(\epsilon)$. Now, let $\Gamma$ be any diagonal matrix with strictly positive elements. Define

$$S(\epsilon) = T(\epsilon)\Gamma T'(\epsilon). \tag{3.8}$$

It is easy to verify by direct computation that $S(\epsilon)$ as given by (3.8) satisfies (3.5). Furthermore, it is clearly symmetric and positive definite.

Analyticity of $S(\epsilon)$ follows directly from the fact that the eigenvectors of $\Phi(\epsilon)$ are analytic functions of $\epsilon$. The following theorem (Chow and Hale, [20, ch. 19]) establishes this fact.

*Theorem 3.5:* Let $\Phi(\epsilon)$: $R^n \to R^n$ be an analytic function of the parameter $\epsilon$. If $\lambda_0$ is a simple eigenvalue with corresponding eigenvector $t_0$ of unit norm of $\Phi(0)$, then there is a constant $\delta > 0$ such that for $|\epsilon| < \delta$, there are unique analytic functions $\lambda(\epsilon)$ with $\lambda(0) = \lambda_0$ and $t(\epsilon)$ with $t(0) = t_0$, such that $\lambda(\epsilon)$ is a simple eigenvalue of $\Phi(\epsilon)$ and $t(\epsilon)$ the corresponding eigenvector of unit norm.

We are now in a position to develop $S(\epsilon)$ as a power series in $\epsilon$. $S(\epsilon)$ can be expressed in the form

$$S(\epsilon) = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \cdots. \tag{3.9}$$

Since $S(\epsilon)$ is symmetric, each $S_i$, $i = 1, 2, \cdots$ must be symmetric. Using the requirement that $S(\epsilon)$ as represented in (3.9) must symmetrize $M^{-1}(U + \epsilon F)$ for all $\epsilon$, $|\epsilon| < \epsilon_0$ we obtain the symmetry conditions

$$S_0 M^{-1}U = UM^{-1}S_0 \tag{3.10a}$$

$$\{S_i M^{-1}U + S_{i-1}F\} = \{S_i M^{-1}U + S_{i-1}F\}', \qquad i \geqslant 1. \tag{3.10b}$$

It follows immediately from (3.10a) and the symmetry of $U$ that we may take $S_0 = M$. Equation (3.10b) can be rewitten

$$UM^{-1}S_i - S_i M^{-1}U = \Delta_i \tag{3.11}$$

where

$$\Delta_i \triangleq S_{i-1}F - F'S_{i-1}. \tag{3.12}$$

Note that both sides of (3.11) are antisymmetric. Recall that $M^{-1}U$ has distinct eigenvalues and can be diagonalized by a matrix $T$, so that

$$M^{-1}U = T\Lambda T^{-1} \tag{3.13}$$

where

$$\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n = 0).$$

Using (3.13), we can reduce (3.11) to the form

$$\Lambda\tilde{S}_i - \tilde{S}_i\Lambda = \tilde{\Delta}_i \tag{3.14}$$

where

$$S_i = T\tilde{S}_i T', \qquad \Delta_i = T\tilde{\Delta}_i T'. \tag{3.15}$$

The $k, l$ element of $\tilde{S}_i$ is given by

$$(\lambda_k - \lambda_l)[\tilde{S}_i]_{kl} = [\tilde{\Delta}_i]_{kl}. \tag{3.16}$$

Note that $\tilde{S}_i$ is symmetric since $\tilde{\Delta}_i$ is antisymmetric. Let $\Phi_i$ denote the particular solution

$$[\Phi_i]_{kl} = \begin{cases} [\tilde{\Delta}_i]_{kl}/(\lambda_k - \lambda_l), & k \neq l \\ 0, & k = l. \end{cases} \tag{3.17}$$

Then

$$\tilde{S}_i = \Phi_i + \Gamma_i \tag{3.18}$$

where $\Gamma_i$ is an arbitrary diagonal matrix. It follows that

$$S_0 = M \tag{3.19a}$$

$$S_i = T(\Phi_i + \Gamma_i)T', \qquad i \geqslant 1. \tag{3.19b}$$

Equation (3.19) explicitly provides the elements $S_i$ of the expansion (3.9).

We are now in a position to prove the following theorem.

*Theorem 3.6:* If the equilibrium manifold of system (3.1) is strongly stable, then there exists $S(\epsilon)$ and $\epsilon_0$ such that the function

$$V(\dot{x}, x, \epsilon) = \dot{x}'S(\epsilon)\dot{x} + xS(\epsilon)M^{-1}(U + \epsilon F)x \tag{3.20}$$

is a Lyapunov function for (3.3) for all $\epsilon$, $|\epsilon| < \epsilon_0$.

*Proof:* Assume $\epsilon_0$ is chosen so that $S(\epsilon)$ exists for all $|\epsilon| < \epsilon_0$. If (3.3) is premultiplied by $\dot{x}'\ S(\epsilon)M^{-1}$ and integrated by parts it is easy to prove that (3.20) is a first integral of (3.3) so that $\dot{V}(\dot{x}, x, \epsilon) = 0$ along trajectories of (3.3). Moreover, since $\dot{x}'S(\epsilon)\dot{x} > 0$ for $\dot{x} \neq 0$, it remains only to show that $x'S(\epsilon)M^{-1}(U + \epsilon F)x \geqslant 0$ where the equality holds only on the subspace of translational symmetry. Since $U, F$ both have translational symmetry it follows that $x'S(\epsilon)M^{-1}(U + \epsilon F)x = 0$ for $x \in \text{Span}\ \{\mathbf{1}\}$ and sufficiently small $\epsilon$.

Furthermore, we have

$$x'S(\epsilon)M^{-1}(U + \epsilon F)x = x'Ux + 0(\epsilon)$$

so that $x'S(\epsilon)M^{-1}(U + \epsilon F)x > 0$ for $x$ not in Span $\{\mathbf{1}\}$ and sufficiently small $\epsilon$.

*Remark:* $V(\dot{x}, x, \epsilon)$ is an energy-like Lyapunov function in the sense that $V(\dot{x}, x, \epsilon)$ depends smoothly on $\epsilon$ and $V(\dot{x}, x, 0)$ corresponds to the energy (modulo a factor of 1/2) of the conservative system. Moreover, $V(\dot{x}, x, \epsilon)$ is the Jacobi first integral corresponding to the Lagrange system associated with (3.3). For a further elaboration of this point see [15], [16].

## IV. Systems with Large Transfer Conductances and Dissipation

It follows from Theorem 3.4 that the concept of strong stability and the results of Section III regarding existence of a local Lyapunov function apply to any stable, undamped power system even with large transfer conductances. This follows from the fact that the system of (3.4) can be converted to that of (3.1) by pre-multiplying (3.4) by $SM^{-1}$ where $S$ is the symmetrizing matrix of Theorem 3.4.

We will apply the same procedure to the damped system

$$M\ddot{x} + D\dot{x} + Kx = 0 \tag{4.1}$$

or equivalently

$$\ddot{x} + M^{-1}D\dot{x} + M^{-1}Kx = 0 \tag{4.2}$$

where $D' = D > 0$, and $K$ has positive real eigenvalues except for precisely one zero eigenvalue corresponding to the translational symmetry.

*Definition:* The system (4.1) or (4.2) is similar to a symmetric system of there exists a real transformation of coordinates $y = Tx$, $|T| \neq 0$, such that the equations of motion have real symmetric coefficients in the new coordinate system.

The following theorem was given by Inman [25]:

*Theorem 4.1:* The system (4.2) is similar to a symmetric system iff $M^{-1}D$ and $M^{-1}K$ have a common symmetrizing matrix.

For a system similar to a symmetric system many well-known results apply. Let us state just one.

*Theorem 4.2:* If the system (4.2) is similar to a symmetric system, then the equilibrium manifold $\hat{M} = \{(\dot{x}; x) | \dot{x} = 0,\ x \in \text{Span}\ \{\mathbf{1}\}\}$ asymptotically stable.

*Proof:* The proof is a straightforward application of Theorem 2.2. Let $S$ be the common symmetrizing matrix of Theorem 4.1 and define the candidate Lyapunov function

$$V(\dot{x}, x) = \dot{x}'S\dot{x} + x'(SM^{-1}K)x. \tag{4.3}$$

Direct calculation leads to

$$\dot{V} = -2\dot{x}'(SM^{-1}D)\dot{x}. \tag{4.4}$$

Clearly, $V$ satisfies the positive definiteness requirements of Theorem 2.2, where $\hat{M}$ is the invariant set. Moreover, $\dot{V} \leqslant 0$ and the inequality holds only for $\dot{x} = 0$. But all solutions satisfying $\dot{x} = 0$ lie entirely in $\hat{M}$. This completes the proof.

One special case of interest is when $M^{-1}D$ and $M^{-1}K$ commute. This includes the case of uniform damping, that is $M^{-1}D = \alpha I$, $\alpha$ a positive scalar. When $M^{-1}D$ and $M^{-1}K$ commute, they have a common set of eigenvectors (Gantmacher [24]) so that

$$\begin{aligned} M^{-1}D &= T^{-1}\Sigma T, \qquad \Sigma = \text{diag}(\sigma_1, \cdots, \sigma_n) \\ M^{-1}K &= T^{-1}\Lambda T, \qquad \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n) \end{aligned} \tag{4.6}$$

and thus the matrix $S = T'T$ is a common symmetrizing matrix. It follows that the conclusions of Theorem 4.2 apply. Thus a power system with transfer conductances which is stable in the absence of damping is asymptotically stable in the presence of commutative (specially uniform) damping.

Suppose, however, that (4.2) is not similar to a symmetric system. We can still utilize the symmetrizing matrix, $S$, for $M^{-1}K$ and rewrite (4.2) as

$$S\ddot{x} + (C + G)\dot{x} + SM^{-1}Kx = 0 \tag{4.7}$$

where

$$\begin{aligned} C &= \tfrac{1}{2}\left[(SM^{-1}D) + (SM^{-1}D)'\right] \\ G &= \tfrac{1}{2}\left[(SM^{-1}D) - (SM^{-1}D)'\right] \end{aligned} \tag{4.8}$$

so that $C$ is symmetric and $G$ is antisymmetric.

We can characterize the stability of (4.7) in terms of the matrix $C$.

*Theorem 4.3:* If $C \geqslant 0$ and the $n^2$ by $n$ matrix

$$\begin{bmatrix} C \\ C(SM^{-1}K) \\ \vdots \\ C(SM^{-1}K)^{n-1} \end{bmatrix}$$

has rank $n$, then the equilibrium manifold is asymptotically stable.

*Remark:* This theorem extends a result of Walker and Schmitendorf [27] to the case $G \neq 0$. It should be viewed as yet another variant of the classical Kelvin–Tait–Chetaev theorem [28].

*Proof:* Once again we use the Lyapunov function defined in (4.3). A simple computation shows that its time derivative along trajectories of (4.7) is

$$\dot{V} = -2\dot{x}'C\dot{x} \tag{4.9}$$

which is negative semidefinite. Thus, by Theorem 2.2, it is now necessary to show that any solution of (4.7) satisfying $\dot{x}'C\dot{x} = 0$ lies in the equilibrium manifold $\hat{M}$. We will do this by showing that under the hypothesis of the theorem only the trivial solution of (4.7) can satisfy this condition.

Assume that

$$\dot{x}'C\dot{x} = 0 \tag{4.10}$$

on some nontrivial time interval $(t_0, t_1)$. Pre-multiply (4.7) by $\dot{x}'$ to obtain

$$\dot{x}'S\ddot{x} + \dot{x}'SM^{-1}Kx = 0 \tag{4.11}$$

on $(t_0, t_1)$. Now choose $t$, $t_0 < t < t_1$, and integrate by parts over $(t_0, t)$ to reveal

$$\dot{x}'S\dot{x} + x'SM^{-1}Kx = \int^t \dot{x}'S\ddot{x}\,dt + \int^t \dot{x}'SM^{-1}K\dot{x}\,dt. \tag{4.12}$$

The left-hand side is readily identifiable as $V$. Thus a necessary condition that $V$ is nonincreasing is

$$\dot{x}'SM^{-1}K\dot{x} = 0. \tag{4.13}$$

It is easy to prove that this condition is sufficient as well. Since $SM^{-1}K$ has a one-dimensional null space spanned by the vector $\mathbf{1}$, the only solutions of (4.7) satisfying (4.13) are of the form $\mathbf{1}\phi(t)$ where $\phi(t)$ is a scalar function of $t$. Direct substitution into (4.7) and pre-multiplication by $\mathbf{1}'$ leads to the conclusions that $\ddot{\phi} = 0$, or equivalent $\ddot{x} = 0$. It follows that the right-hand side of (4.12) is constant. Thus a solution of (4.7) satisfies (4.10) iff it also satisfies (4.13).

Note that (4.10) and (4.13) can be satisfied simultaneously iff there exists a nontrivial vector $q$ in the null spaces of both $C$ and $SM^{-1}K$. The conditions of the theorem preclude this as will be proved in the following. Assume that there exists a nontrivial $q$ in the null space of $SM^{-1}K$. We will show that it can not lie in the null space of $C$. Write the sequence of relations

$$\begin{aligned} Cq &= Cq \\ C(SM^{-1}K)q &= 0 \\ &* \\ &* \\ C(SM^{-1}K)^{n-1}q &= 0. \end{aligned} \tag{4.14}$$

Since the coefficient matrix on the left has full rank by hypothesis, it has a left inverse. Let $\Sigma$ denote the first $n$ columns of the left inverse. Then

$$q = \Sigma Cq. \tag{4.15}$$

Clearly $Cq \neq 0$. It follows that (4.7) does not possess a nontrivial solution satisfying (4.10).

If $C$ is indefinite, then the equilibrium manifold may be unstable. The significance of this result arises from the fact that the definiteness properties of $C$ do not directly follow from those of $D$. It is true that if $D$ has positive real eigenvalues then so does $C + G$. However, $C$ may not have positive real eigenvalues and it is $C$ which determines the stability of (4.7). This observation was made by Huseyin and Hagedorn [26]. The important implication is that a power system, with transfer conductances, which is stable in the absence of dissipation may be destabilized by the addition of dissipation.

In the absence of transfer conductances, the energy function represents the "perfect" Lyapunov function in the sense that it globally characterizes the stability properties of the system. The energy function itself precisely determines the domain of stability of the stable equilibrium manifold. It is not known whether a global counterpart to the energy function exists in the presence of transfer conductances. Consideration of this question is well beyond the scope of the present paper. However, in this regard, we can give an interpretation of a Lyapunov function proposed by DiCaprio [29], [30]. In view of the remark following Theorem 3.6, it is reasonable to conjecture that if a global energy-like potential function exists for a system with transfer conductances its local character will be that of (4.3). We can easily define a class of candidate Lyapunov functions which possess the following two properties: 1) they are locally equivalent to (4.3), and 2) they reduce globally to the conservative system energy function in the absence of transfer conductances.

First, the function $f(\delta)$ (recall the nonlinear model in Section II) can be nonuniquely separated

$$f(\delta) = f_1(\delta) + f_2(\delta) \tag{4.16}$$

so that $f_1$ is integrable and $f_2$ is not necessarily so, $f_1$ and $f_2$ have the translational symmetry property of $f$, and $f_1$ reduces to $f$ in the absence of transfer conductances. One simple choice for $f_1$ is obtained from $f$ by setting the transfer conductances equal to zero. There are infinitely many others. One might conjecture that the best choice for $f_1$ is that which renders the Jacobian of $f_2$ antisymmetric. It is not known whether such a decomposition exists. Let $U_1(\delta)$ represent a potential function from which $f_1$ is derivable. As before, denote an equilibrium point of interest by $\delta^*$ and define the potential function $V_1$:

$$V_1(\dot{\delta}, \delta) = \dot{\delta}'M\dot{\delta} + U_1(\delta) - U_1(\delta^*) + \{f_2(\delta^*) - P\}'(\delta - \delta^*). \tag{4.17}$$

It follows directly from the construction of $U_1$ that $V_1$ has the desired global property, i.e., it reduces to the conservative system energy function (modulo a factor of $1/2$) in the absence of transfer conductances. However, in general, it is not locally equivalent to (4.3). This is easily observed by rewriting (4.17) in local coordinates, $x = \delta - \delta^*$

$$\begin{aligned} V_1 &= x'Mx + U_1(x + \delta^*) - U_1(\delta^*) + \{f_2(\delta^*) - P\}'x \\ &= x'Mx + x'Ux + 0(\|x\|^3) \end{aligned} \tag{4.18}$$

where $2U = \partial U_1(\delta^*)/\partial\delta$. A simple remedy is to replace the quadratic terms in (4.18) by the right-hand side of (4.3) to obtain

$$V(\dot{\delta},\delta) = V_1(\dot{\delta},\delta) + \dot{\delta}'\{S - M\}\dot{\delta} + (\delta - \delta^*)'\{SM^{-1}K - U\}(\delta - \delta^*) \quad (4.19)$$

where $K$ is defined, as before, by (2.24), and it is evident from our previous results that $V$ collapses to $V_1$ in the absence of transfer conductances, so that the desired global property is preserved. The class of Lyapunov functions defined by (4.17) and (4.19) includes the Lyapunov function proposed by DiCaprio. We await the publication of DiCaprio's experience in the application of this Lyapunov function in order to judge its practical merits.

## V. Conclusions

The main result of this paper is the proof that a local energy-like Lyapunov function exists for almost all stable electric power systems governed by the standard model and including those with nonzero transfer conductances. Moreover, we have given a simple characterization of the exceptional stable systems and noted that the conclusions of Narasinhamurthi asserting nonexistence of energy-like Lyapunov functions is based on such an exceptional system.

The distinguishing feature between those systems which possess an energy-like Lyapunov function and those that do not is the property of strong stability. We have introduced a notion of strong stability especially tailored for the power system stability problem. It is shown that strong stability is a generic property of stable power systems, and that it is a necessary and sufficient condition for the existence of an energy-like Lyapunov function. In broad terms, strong stability is the ability of a stable power system to remain stable under sufficiently small, but arbitrary perturbations in the reduced bus admittance matrix.

The effect of dissipation on the stability of systems with transfer conductances has been examined. It has been shown that uniform damping belongs to a larger class of damping models which obey a certain commutativity property and that dissipation of this type always has a stabilizing effect of the system. However, it has also been noted that arbitrary damping can destabilize systems with transfer conductances. This is an important observation because damping in power systems arises in small amounts from many different sources. There is no universally accepted model of dissipation and the frequently used uniform damping model is simply a matter of analytical convenience.

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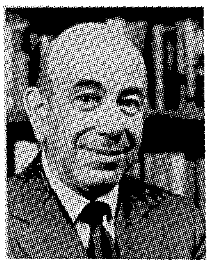

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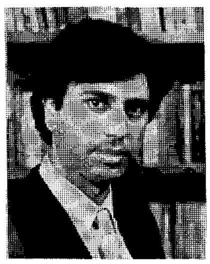

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